VARIATIONAL METHODS FOR NONLINEAR EQUATIONS OF MOTION OF SHELLS

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A variational principle is developed for dynamic problems with initial conditions of geometrically nonlinear theory of elasticity.

Approximations are introduced for displacements, deformations, stresses, velocities of displacements and impulses and the complete variational principle is derived for the simplest variant of nonlinear elastic shell theory of the Timoshenko-type and for the corresponding adjoint problem.

The nonlinear theory of dynamics which takes into account the deformation of shear and the inertia of rotation was examined in papers [1 to 4] for plates and in papers [5 to 8] for shells.

Variational principles for dynamic problems with initial conditions from linear theory of elasticity were developed in papers [9 and 10], and for linear theory of the Timoshenko-type for elastic shells in paper [11].

1. Variational principle of nonlinear theory of elasticity. The nonlinear theory of elasticity can be presented in the form of a variational problem which requires the finding of the steady-state value of the following functional:

$$J = \int_V \langle -{}^1/_2 E^{ikjl} \varepsilon_{ik} * \varepsilon_{jl} + \tau^{ik} * [\varepsilon_{ik} - {}^1/_2 (e_{ik} + e_{ki} + e_i^{\cdot j} e_{kj})] + \sigma^{ik} * (e_{ik} - \overline{\nabla}_i u_k) -$$

$$- {}^1/_2 \rho v_i * v^i + \pi^i * (s_i - u^{\cdot}_i) + X^i * u_i - [u_i(P,0) - u_i^\circ] \pi^i (P, \tau) + \pi_i^\circ u^i (P, \tau) \rangle' dV +$$

$$+ \int_{S_1} \langle Q^i * u_i \rangle' dS + \int_{S_2} \langle (u_k - U_k) * \sigma^{ik} n_i \rangle' dS \qquad (1.1)$$

$$F * G = \int_0^\tau F(P,t) G(P, \tau - t) dt, \qquad \langle L * M \rangle' = \langle L \rangle' * M + L * \langle M \rangle' \qquad (1.2)$$

$$\langle L(f_1, \ldots, f_n) \rangle' = \sum_{s=1}^n \left[\frac{\partial L}{\partial f_s} \right]_* f_s'$$

Here $f_1, \ldots, f_n$ are functions and their derivatives, encountered in the functional (1.1); the prime designates a definite class of values and

$$[F(P,t)]_* = F(P, \tau - t) \qquad (1.3)$$

In functional (1.1) two systems of quantities appear as variable quantities

$$W = \{u_i, e_{ik}, s_i, \varepsilon_{ik}, \tau^{ik}, \pi^i, \sigma^{ik}\} \qquad (1.4)$$

and the same quantities with primes, i.e. $\langle W \rangle'$.

Their physical significance becomes apparent from steady-state conditions of the functional (1.1). These conditions can be divided into two groups.

For the initial system W the following system of Eqs. is obtained:

equations of motion

$$\overline{\nabla}_k \sigma^{ki} + X^i - \pi^{i\cdot} = 0, \qquad P \in V, \qquad 0 < t < \tau \tag{1.5}$$

kinematic relationships

$$e_{ik} = \overline{\nabla}_i u_k, \qquad s_i = u_i^{\cdot}, \qquad \varepsilon_{ik} = {}^1/_2 (e_{ik} + e_{ki} + e_{i\cdot}^{\cdot j} l_{kj}) \tag{1.6}$$

relationships of elasticity

$$\tau^{ik} = E^{ikjl} \varepsilon_{jl}, \qquad \pi_i = \rho s_i, \qquad \sigma^{ik} = \tau^{ij} (\delta_j^k + e_j^{\cdot k}) \tag{1.7}$$

boundary conditions

$$\sigma^{ik} n_i = Q^k, \quad P \in S_1 \quad (0 < t < \tau) \; u_i = U_i, \quad P \in S_2 \quad (0 < = < \tau) \tag{1.8}$$

initial conditions

$$u_i = u_i^\circ, \qquad \pi_i = \pi_i^\circ \qquad P \in V, \; t = 0 \tag{1.9}$$

If equations and relationships (1.5) to (1.9) are briefly denoted by

$$\{(1.5) - (1.9)\} \equiv \{L = 0\} \tag{1.10}$$

then the second group of conditions for the steady-state of functional (1.1) is represented in the form

$$\langle L \rangle' = 0 \tag{1.11}$$

Relationships (1.11) determine the quantities $\langle W \rangle'$ and form the so-called adjoint problem.

Thus, equations and relationships of nonlinear theory of elasticity (1.5) to (1.9) together with the corresponding relationships of the adjoint problem (1.11) can be formulated in the form of the variational principle

$$\delta J = 0 \tag{1.12}$$

2. Geometry of the shell and hypotheses of the theory. Let us examine a shell of constant thickness h and let us take advantage of the usual curvilinear coordinates of the shell with a radius-vector of the undeformed body

$$\mathbf{R} = \mathbf{r}(x^1, x^2) + x^3 \mathbf{n}(x^1, x^2) \tag{2.1}$$

where $\mathbf{r}$ and $\mathbf{n}$ are the radius vector and the unit vector of the normal to the mean surface of the undeformed shell.

Let $a_{\alpha\beta}$ and $b_{\alpha\beta}$ be the tensors of the first and second quadratic form of the mean surface, i.e.

$$a_{\alpha\beta} = \mathbf{r}_\alpha \mathbf{r}_\beta, \quad b_{\alpha\beta} = -\mathbf{r}_\alpha \mathbf{n}_\beta, \left(\mathbf{r}_\alpha = \frac{\partial \mathbf{r}}{\partial x^\alpha}, \quad \mathbf{n}_\alpha = \frac{\partial \mathbf{n}}{\partial x^\alpha} \right) \quad (\alpha, \beta = 1, 2) \tag{2.2}$$

and ∇_α is the symbol for covariant differentiation in the metric of $a_{\alpha\beta}$.

Correspondingly, components of the metric tensor of the undeformed three-dimensional body

$$g_{ik} = \mathbf{R}_i \mathbf{R}_k, \qquad \mathbf{R}_i = \frac{\partial \mathbf{R}}{\partial x^i} \tag{2.3}$$

are expressed through the metric tensor of the shell

$$g_{\alpha\beta} = \mu_\alpha^{\gamma} \mu_\beta^{\delta} a_{\gamma\delta}, \quad g_{\alpha 3} = 0, \quad g_{33} = 1, \quad (\mu_\beta^{\alpha} = \delta_\beta^{\alpha} - x^3 b_\beta^{\alpha}) \tag{2.4}$$

If the displacement vector is written in the form

$$\mathbf{u} = u_\alpha^* \mathbf{r}^\alpha + u_3^* \mathbf{n} \tag{2.5}$$

then the following relationships apply

$$u_\alpha = \mu_\alpha^{\beta} u_\beta^*, \qquad u_3 = u_3^*, \qquad \overline{\nabla}_\beta u_\alpha = \mu_\alpha^{\gamma} (\nabla_\beta u_\gamma^* - b_{\beta\gamma} u_3^*)$$

$$\overline{\nabla}_3 u_\alpha = \mu_\alpha^{\ \beta} \frac{\partial u_\beta^*}{\partial x^3}, \qquad \overline{\nabla}_\alpha u_3 = \nabla_\alpha u_3^* + b_\alpha^{\ \beta} u_\beta^*, \qquad \overline{\nabla}_3 u_3 = \frac{\partial u_3^*}{\partial x^3} \tag{2.6}$$

In order to derive the functional of the theory of shells from the functional (1.1) for the three-dimensional elastic body it is necessary to utilize a certain system of hypotheses. In order to obtain the simplest variant of nonlinear theory of the Timoshenko-type we shall start from the following system of hypotheses.

1. The quantity $x^3 b_{\alpha\beta}$ is considered sufficiently small to be neglected in comparison to one ($x^3 b_{\alpha\beta} \ll 1$), i.e. we consider that the metric does not change with the thickness of the shell

$$\mu_\beta^{\ \alpha} = \delta_\beta^{\ \alpha} \tag{2.7}$$

2. We shall assume that the distribution of variable quantities over the thickness of the shell corresponds to relationships

$$u_\alpha^* = v_\alpha + x^3 \varphi_\alpha, \qquad u_3^* = w \tag{2.8}$$

$$s_\alpha = \theta_\alpha + x^3 \alpha_\alpha, \qquad s_3 = \beta \tag{2.9}$$

$$e_{\alpha\beta} = \eta_{\alpha\beta} + x^3 \varkappa_{\alpha\beta}, \quad e_{\alpha 3} = \omega_\alpha + x^3 \mu_\alpha, \quad e_{3\alpha} = \iota_\alpha, \quad e_{33} = 0 \tag{2.10}$$

$$\varepsilon_{\alpha\beta} = \gamma_{\alpha\beta} + x^3 \xi_{\alpha\beta}, \qquad \varepsilon_{\alpha 3} = f(x^3)\, \vartheta_\alpha, \qquad \varepsilon_{33} = \zeta + \chi x^3 \tag{2.11}$$

$$\pi_\alpha = \Theta_\alpha + x^3 A_\alpha, \qquad \pi_3 = \Lambda \tag{2.12}$$

$$\sigma^{\alpha\beta} = \frac{1}{h} t^{\alpha\beta} + \frac{12 x^3}{h^3} m^{\alpha\beta}, \qquad \sigma^{\alpha 3} = \frac{1}{h} f(x^3)\,(k^\alpha + x^3 l^\alpha) \tag{2.13}$$

$$\sigma^{3\alpha} = \frac{1}{h} f(x^3)\, q^\alpha, \qquad \sigma^{33} = 0$$

$$\tau^{\alpha\beta} = \frac{1}{h} T^{\alpha\beta} + \frac{12 x^3}{h^3} M^{\alpha\beta}, \quad \tau^{\alpha 3} = \frac{1}{h} f(x^3)\, N^\alpha, \quad \tau^{33} = 0 \tag{2.14}$$

and that analogous equations also exist for the quantity $\langle W \rangle'$.

Here $f(x^3)$ is a given even function satisfying the conditions

$$\int_{-h/2}^{h/2} f(x^3)\,\{1,\ (x^3)^2,\ f\}\, dx^3 = h\,\{1,\ k^*,\ k\} \tag{2.15}$$

3. In nonlinear terms entering into the functional (1.1) $e_{k\cdot}^{\ \cdot j} e_{kj}$, $\tau^{ij} e_{i\cdot j}^{\ \ \cdot k}$ we shall neglect after substitution (2.10) and (2.14) terms with the factor $(x^3)^2$.

From relationships (2.13) and (2.14) it follows that

$$\{t^{\alpha\beta},\ m^{\alpha\beta},\ k^\alpha,\ q^\alpha\} = \int_{-h/2}^{h/2} \{\sigma^{\alpha\beta},\ x^3\sigma^{\alpha\beta},\ \sigma^{\alpha 3},\ \sigma^{3\alpha}\}\, dx^3 \tag{2.16}$$

$$\{T^{\alpha\beta},\ M^{\alpha\beta},\ N^\alpha\} = \int_{-h/2}^{h/2} \{\tau^{\alpha\beta},\ x^3\tau^{\alpha\beta},\ \tau^{\alpha 3}\}\, dx^3 \tag{2.17}$$

Let the following notations be introduced for external loads

$$\{p_\alpha,\ p,\ m_\alpha\} = \{Q_\alpha,\ Q_3,\ x^3 Q_\alpha\}\Big|_{x^3=-h/2}^{x^3=h/2} \tag{2.18}$$

$$\{P^\alpha,\ K,\ K^\alpha\} = \int_{-h/2}^{h/2} \{Q_\alpha,\ Q,\ x^3 Q_\alpha\}\, dx^3 \tag{2.19}$$

and also analogous notations for quantities with primes.

Further, we shall assume that volume forces are absent, i.e.

$$X_i = 0, \qquad X_i' = 0 \tag{2.20}$$

We shall introduce also the following notations:

$$\{V_\alpha, V_3, \Phi_\alpha\} = \frac{1}{h} \int_{-h/2}^{h/2} \left\{U_\alpha, fU_3, \frac{12x^3}{h^2} U_\alpha\right\} dx^3 \tag{2.21}$$

$$\{v_\alpha^\circ, w^\circ, \varphi_\alpha^\circ\} = \frac{1}{h} \int_{-h/2}^{h/2} \left\{u_\alpha^\circ, u_3^\circ, \frac{12x^3}{h^2} u_\alpha^\circ\right\} dx^3 \tag{2.22}$$

$$\{\Theta_\alpha^\circ, \Lambda^\circ, A_\alpha^\circ\} = \int_{-h/2}^{h/2} \{\pi_\alpha^\circ, \pi_3^\circ, x^3\pi_\alpha^\circ\} dx^3 \tag{2.23}$$

and the analogous notation for quantities with primes.

3. The complete variational principle. Now we introduce Expressions (2.6) to (2.14) into the functional (1.1). Taking into consideration Expressions (2.15) to (2.23) and the third hypothesis and performing integration over the coordinate x^3 we have

$$J = \int_V \Big\langle -\frac{1}{2} hE^{\alpha\beta\gamma\delta}\left(\gamma_{\alpha\beta} * \gamma_{\gamma\delta} + \frac{h^2}{12}\xi_{\alpha\beta} * \xi_{\gamma\delta}\right) - h\lambda a^{\alpha\beta}\left(\zeta * \gamma_{\alpha\beta} + \frac{h}{12}\chi * \xi_{\alpha\beta}\right) -$$

$$- 2\mu hk\vartheta_\alpha * \vartheta^\alpha - {}^1/_2(\lambda + 2\mu)\, h\left(\zeta * \zeta + {}^1/_{12}h^2\chi * \chi\right) T^{\alpha\beta} * [\gamma_{\alpha\beta} - {}^1/_2\ (\eta_{\alpha\beta} + \eta_{\beta\alpha} + \eta_{\alpha}^{\cdot\gamma}\eta_{\beta\gamma} +$$

$$+ \omega_\alpha\omega_\beta)] + M^{\alpha\beta} * [\xi_{\alpha\beta} - {}^1/_2\,(\varkappa_{\alpha\beta} + \varkappa_{\beta\alpha} + \eta_{\alpha}^{\cdot\gamma}\varkappa_{\beta\gamma} + \eta_{\beta}^{\cdot\gamma}\varkappa_{\alpha\gamma} + \omega_\alpha\mu_\beta + \omega_\beta\mu_\alpha)\,] +$$

$$+ 2N^\alpha * [k\vartheta_\alpha - {}^1/_2(\omega_\alpha + i_\alpha + \eta_{\alpha}^{\cdot\beta} i_\beta)] - {}^1/_2\,\rho h\theta_\alpha * \theta^\alpha - {}^1/_{24}\rho h^3\alpha_\gamma * \alpha^\gamma -$$

$$- {}^1/_2\rho h\beta * \beta + t^{\alpha\beta} * (\eta_{\alpha\beta} - \nabla_\alpha v_\beta + b_{\alpha\beta}w) + m^{\alpha\beta} * (\varkappa_{\alpha\beta} - \nabla_\alpha\varphi_\beta) +$$

$$+ k_\alpha * (\omega_\alpha - \nabla_\alpha w - b_{\alpha\beta}v^\beta) + k{*}l^\alpha * (\mu_\alpha - b_{\alpha\beta}\varphi^\beta) + q^\alpha * (i_\alpha - \varphi_\alpha) + h\Theta^\alpha * (\theta_\alpha - v_\alpha^{\cdot}) +$$

$$+ {}^1/_{12}h^3A^\beta * (\alpha_\beta - \varphi_\beta^{\cdot}) + h\Lambda * (\beta - w^{\cdot}) - h\,[v_\alpha(P, 0) - v_\alpha^\circ]\,\Theta^\alpha(P,\ t) -$$

$$- {}^1/_{12}\,h^3[\varphi_\alpha(P, 0) - \varphi_\alpha^\circ]\,A^\alpha(P, \tau) - h\,[w(P,\ 0) - w^\circ]\,\Lambda(P,\ \tau) + \Theta_\alpha^\circ v^\alpha(P,\ \tau) +$$

$$+ \Lambda^\circ w(P, \tau) + A_\alpha^\circ\varphi^\alpha(P,\ \tau) + p^\alpha * v_\alpha + p * w + m^\alpha * \varphi_\alpha\Big\rangle' dV +$$

$$+ \int_{C_1} \langle P^\alpha * v_\alpha + K * w + K^\alpha * \varphi_\alpha\rangle' dC +$$

$$+ \int_{C_2} \langle (v_\alpha - V_\alpha) * t^{\alpha\beta}n_\beta + (w - W) * k^\alpha n_\alpha + (\varphi_\alpha - \Phi_\alpha) * m^{\alpha\beta}n_\beta\rangle' dC \tag{3.1}$$

$$E^{\alpha\beta\gamma\delta} = \lambda a^{\alpha\beta}a^{\gamma\delta} + 2\mu a^{\alpha\gamma}a^{\beta\delta} \tag{3.2}$$

Here S is the mean surface of the undeformed shell, n_α are the components of the unit vector of the normal to the contour of the undeformed shell C.

Steady-state conditions of functional (3.1) will be two systems of equations with boundary conditions and initial conditions for the basic problem and for the adjoint problem. The first system has the following form:

equations of motion

$$\nabla_\alpha t^{\alpha\beta} - b_\alpha^{\ \beta}k^\alpha - h\Theta^{\beta\cdot} + p^\beta = 0, \qquad P \in S; \quad 0 < t < \tau$$

$$\nabla_\alpha k^\alpha + b_{\alpha\beta}t^{\alpha\beta} - h\Lambda^{\cdot} + p = 0, \ \nabla_\alpha m^{\alpha\beta} - q^\beta - k{*}b_\alpha^{\ \beta}l^\alpha - \frac{h^2}{12}A^{\beta\cdot} + m^\beta = 0 \tag{3.3}$$

kinematic relationships

$$\eta_{\alpha\beta} = \nabla_\alpha v_\beta - b_{\alpha\beta}u, \quad \varkappa_{\alpha\beta} = \nabla_\alpha\varphi_\beta, \quad \omega_\alpha = \nabla_\alpha w + b_{\alpha\beta}v^\beta, \quad \mu_\alpha = b_{\alpha\beta}\varphi^\beta, \quad i_\alpha = \varphi_\alpha$$
$$\theta_\alpha = v_\alpha^{\cdot}, \qquad \beta = w^{\cdot}, \qquad \alpha_\gamma = \varphi_\gamma^{\cdot} \tag{3.4}$$

$$\gamma_{\alpha\beta} = {}^1/_2\,(\eta_{\alpha\beta} + \eta_{\beta\alpha} + \eta_{\alpha\cdot}^{\cdot\gamma}\eta_{\beta\gamma} + \omega_\alpha\omega_\beta)$$
$$\xi_{\alpha\beta} = {}^1/_2\,(\varkappa_{\alpha\beta} + \varkappa_{\beta\alpha} + \eta_{\alpha\cdot}^{\cdot\gamma}\varkappa_{\beta\gamma} + \eta_{\beta\cdot}^{\cdot\gamma}\varkappa_{\alpha\gamma} + \omega_\alpha\mu_\beta + \omega_\beta\mu_\alpha) \tag{3.5}$$
$$k\vartheta_\alpha = {}^1/_2\,(\omega_\alpha + \iota_\alpha + \eta_{\alpha\cdot}^{\cdot\beta}\iota_\beta)$$

elasticity relationships

$$T^{\alpha\beta} = hE^{\alpha\beta\gamma\delta}\gamma_{\gamma\delta} + h\lambda a^{\alpha\beta}\zeta$$
$$M^{\alpha\beta} = \frac{h^3}{12}E^{\alpha\beta\gamma\delta}\zeta_{\gamma\delta} + \frac{h^3}{12}\lambda a^{\alpha\beta}\chi, \qquad N^\alpha = 2h\mu\vartheta^\alpha, \qquad (\lambda + 2\mu)\,\zeta = \lambda a^{\alpha\beta}\gamma_{\alpha\beta} \tag{3.6}$$
$$(\lambda + \mu)\,\chi = -\lambda a^{\alpha\beta}\xi_{\alpha\beta} \quad \Theta_\alpha = \rho\theta_\alpha, \qquad \Lambda = \rho\beta, \qquad A_\gamma = \alpha_\gamma$$

$$t^{\alpha\beta} = T^{\alpha\gamma}(\delta_\gamma^{\ \beta} + \eta_{\gamma\cdot}^{\cdot\beta}) + N^\alpha\iota^\beta + M^{\alpha\gamma}\varkappa_{\gamma\cdot}^{\cdot\beta}$$
$$m^{\alpha\beta} = M^{\alpha\gamma}(\delta_\gamma^{\ \beta} + \eta_{\gamma\cdot}^{\cdot\beta}), \qquad k^\alpha = N^\alpha + T^{\alpha\beta}\omega_\beta + M^{\alpha\beta}\mu_\beta \tag{3.7}$$
$$k_* l^\alpha = M^{\alpha\beta}\omega_\beta, \qquad q^\alpha = N^\beta(\delta_\beta^{\ \alpha} + \eta_\beta^{\ \alpha})$$

boundary conditions

$$t^{\alpha\beta}n_\alpha = P^\beta, \quad m^{\alpha\beta}n_\alpha = K^\beta, \quad k^\alpha n_\alpha = K, \quad P \in C_1, \quad 0 < t < \tau \tag{3.8}$$
$$v_\alpha = V_\alpha, \quad w = V_3, \quad \varphi_\alpha = \Phi_\alpha, \quad P \in C_2, \quad 0 < t < \tau \tag{3.9}$$

initial conditions

$$v_\alpha = v_\alpha^{\circ}, \quad w = w^{\circ}, \quad \varphi_\alpha = \varphi_\alpha^{\circ}, \quad P \in V, \quad t = 0$$
$$\Theta_\alpha = \Theta_\alpha^{\circ}, \qquad \Lambda = \Lambda^{\circ}, \qquad A_\alpha = A_\alpha^{\circ} \tag{3.10}$$

Let us introduce the notation

$$\{(3.3) - (3.10)\} \equiv \{M = 0\} \tag{3.11}$$

Taking into account Expression (1.2) the remaining conditions for the functional (3.1) being stationary can now be represented in the form

$$\langle M\rangle' = 0 \tag{3.12}$$

Equations and relationships (3.12) form a linear system with respect to quantities with primes. In the adjoint system (3.12) the same equations in quantities with primes correspond to linear Eqs. of system (3.11). The nonlinear relationships (3.5) and (3.7) of system (3.12) are replaced by linear relationships. The coefficients of these relationships are quantities of the basic system. For example, relationship (see 3.5) for the adjoint problem has the form

$$\gamma_{\alpha\beta}' = {}^1/_2\,(\eta_{\alpha\beta}' + \eta_{\beta\alpha}' + [\eta_{\alpha\cdot}^{\cdot\gamma}]_*\eta_{\beta\gamma} + [\eta_{\beta\cdot}^{\cdot\gamma}]_*\eta_{\alpha\gamma}' + [\omega_\alpha]_*\omega_\beta + [\omega_\beta]_*\omega_\alpha) \tag{3.13}$$

It is easy to see that the equations of the adjoint problem are equations of stability for the motion of shells which is determined by relationships (3.3) to (3.10) but progresses in time in the opposite direction in the interval $(0, \tau)$.

From the complete variational principle (1.12), where now the functional J is given through (3.1), we may obtain all other modifications of the principle for nonlinear equations of the theory of shells if certain groups of stationary conditions of this functional are satisfied first.

In conclusion, we note that in case of linearization of equations in the theory of shells, the adjoint problem can be selected to coincide with the basic problem and therefore in variational formulation there is no need to broaden the initial problem because of the adjoint problem [11].

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